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Approximate Solution for The Nonlinear Fractional Coupled Navier–Stokes of The Fluid Model of $2-$Dimensional by Using A Hybrid Technique

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Abstract

The current paper investigates the fractional Navier–Stokes of the fluid model in which the differential is of non-integer order. In addition, some basic definitions are discussed. This research aims to find an approximate solution to non-linear Fractional Coupled Navier–Stokes Equation (FCNSE) in two-dimension by using a hybrid technique. Thus, we propose a hybrid the Shehu transform ($\mathbb{S}$-transform) with the homotopy perturbation method to solve this model. The $\mathbb{S}$-transform with homotopy perturbation is an excellent combination in applied mathematics and engineering that permits in converting FCNSE into algebraic equations . Then, through solving this algebraic equation, it is possible to obtain the unknown function utilizing some modifications with the help of inverse $\mathbb{S}$- transform. For demonstrating the effectiveness and capabilities of the proposed innovation, various illustrative examples were applied.

Keywords: fractional calculus; homotopy perturbation method; Shehu transform; Navier–Stokes.

1 Introduction

Realistic problems related to fractional differential equations are of great significance, where Bilal et al. [6] used the collocation method for solving fractional pantograph differential equation with fractional Taylor series. Albuohimad et al. [2] offered a numerical approach capable of resolving the fractional coupled Korteweg-de Vries equation. Saadatmandi and Dehghan [18] worked on generalizing the legendre matrix of operations to fractional calculus. Under the title of fractional differential equations, Podlubny [16] studied fractional derivatives with some applications. Set et al. [20] created inequalities related to fractional integral. Since fractional differential equations play a crucial part in many applied and theoretical studies, a large number of researchers have tackled the fractional version of various phenomena in nature and biology, such as in [7], "A fractional integral sliding mode control scheme based on Caputo-Fabrizio derivative and Atangana-Balino integral is developed and presented for Stanford robot for path-following tasks". Similarly, "Treatment of childhood hearing loss caused by mumps virus and a unique solution for a specific fractional system for the hearing loss model has been demonstrated" in [13].

Studies have followed where many scholars have focused on that fractional expansions of mathematical models of the order of integers treat natural truth in a very systematic way. Baleanu et al. [4] presented a fractional-degree epidemiological model of childhood diseases using the new fractional derivative approach proposed by Caputo and Fabrizio. Ahmad et al. [1] studied the existence and stability fractional differential system. Dehingia et al. [9] formulated a mathematical model of SARS-CoV-2 under Caputo's fractional degree derivative. Moreover, the function accumulates in a likely manner, especially if they are famous equations such as the Navier–Stokes equations of the fluid [19], and this has many applications in the physical sciences, design, etc. When these equations are mixed together and developed in various sciences, non-linear Fractional Navier–Stokes equations in $2-$dimensional are realized. Accordingly, there will be a possibility of mathematically describing the applied issue and searching for its solution is one of the researchers priorities. Hence, it is noticed that the non-linear Navier–Stokes equations play an essential role in mathematical modeling with the effects of fractional derivative [3], whether they are one or two-dimensional. The original Navier–Stokes models are

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\nabla)\underline{v} = -\frac{1}{\rho}\nabla_P + u\nabla^2\underline{v}, \tag{1}$$

$$\nabla\underline{v} = 0, \tag{2}$$

where ρ, P, u, $\underline{v}$, t are density, pressure, kinematics viscosity, velocity and time respectively.

In this research, equation above can be generalized by replacing the ordinary derivative by a fractional derivative of order $\Im$, $0 < \Im \leq 1$, and develop the Navier–Stokes equations into a coupled Navier–Stokes equations, also from ($1-$dimension) into ($2-$dimension). Therefore, an FCNSE of the following model [8] can be obtained,

$$\begin{aligned} D^{\Im}u + u\frac{\partial u}{\partial \zeta} + u\frac{\partial u}{\partial \xi} &= \rho\left(\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \xi^2}\right) + f_1, \\ D^{\Im}u + u\frac{\partial u}{\partial \zeta} + u\frac{\partial u}{\partial \xi} &= \rho\left(\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \xi^2}\right) + f_2. \end{aligned} \tag{3}$$

For $u = u(\zeta, \xi, t)$, $v = v(\zeta, \xi, t)$, $f_1 = f_1(\zeta, \xi, t)$ and $f_2 = f_2(\zeta, \xi, t)$, the fractional Navier–Stokes equations have been studied by Momani and Odibat [14] by using Homotopy Perturbation Method (HPM). The HPM was first introduced by He [10].

The present article starts with offering few significant definitions, some fundamental ideas for fractional calculus and definition of some important functions. Then, an explanation of the structure employed in solving (3) is presented. This structure includes mixing the Shehu transform [22] with Homotopy perturbation method to solve this type of non-linear fractional coupled Navier–Stokes equations in two dimension and use some modification to get the approximate solutions.

2 Fractional Calculus

In this segment, some of the special functions are introduced. Additionally, some definitions and essential ideas of fractional calculus. Hussein and Jassim [11] obtained geometric properties by studying some classes of polymorphic functions, which are determined by the fractional integration operator. Oldham and Spanier [15] covered the theories and applications of the fractional derivatives and integrals. Also, Tarasov [21] studied applications of the fractional calculus. Kumar and Baleanu [12] focused on the physics applications within the fractional calculus track. They assumed a significant part in the hypothesis of fractional differentiation's. Moreover, these functions and basic are used in developing special formulations that are applicable to fractional differentiation.

2.1 Gama function

The gamma function is the example par excellence for a reasonable extension of scope of a function from integer to real up to imaginary number. The gamma function $\Gamma(\varkappa)$ is defined as,

$$\Gamma(\varkappa) = \int_0^\infty t^{\varkappa-1} e^{-t} dt, \qquad \varkappa > 0.$$

First of all, it is easily shown that for a positive integer $\varkappa$ the gamma function can be represented by,

$$\Gamma(\varkappa) = (\varkappa - 1)!.$$

Therefore,

$$\Gamma(\varkappa + 1) = \varkappa\Gamma(\varkappa).$$

The above equality enable us to calculate, for any positive real $\varkappa$, the function $\Gamma(\varkappa)$ in terms of the fractional part of $\varkappa$.

2.2 The fractional derivative

Typically specialists give the Riemann Liouville(RL) variant of fractional integral definition. However, we are keen on a more valuable meaning of fractional derivatives. The usual formulation of the fractional derivatives have been given in standard references such as Rafeiro and Samko [17], Oldman and Spanier [15], is the Riemann-Liouville definition.

Definition 2.1. *The following form,*

$$D_a^{\Im} u(x) = \frac{1}{\Gamma(\ell - \Im)} \frac{d^\ell}{dx^\ell} \int_a^x (x-t)^{\ell - \Im - 1} u(t) dt, \tag{4}$$

where ℓ is a positive integer number defined by $(\ell - 1 < \Im \leq \ell)$, is called The ***Riemann Liouville derivative*** *with order $\Im > 0$, [16].*

Definition 2.2. *With order $\Im > 0$ of the given function $u(x)$ the following form,*

$$D_a^{\Im} u(x) = I^{\ell-\Im} D_a^{\ell} u(x) = \frac{1}{\Gamma(\ell-\Im)} \int_a^x (x-t)^{\ell-\Im-1} \frac{d^{\ell}}{dx^{\ell}} u(t)dt, \tag{5}$$

*where $(\ell - 1 < \Im \leq \ell)$ is called The **Caputo Derivatives**.*

Lemma 2.1. *If $\ell - 1 < \Im \leq \ell, \ell \in N$, then,*

i. $D^{\Im} I^{\Im} u(x) = u(x)$.

ii. $I^{\Im} D^{\Im} u(x) = u(x) - \sum_{n=0}^{\ell-1} \frac{x^n}{n!} u^{(n)}(0)$, $x > 0$.

Lemma 2.2. *For RL fractional derivative , we have*

- *If $u(x)$ is the unit function, then* $\frac{d^{\Im}[1]}{dx^{\Im}} = \frac{x^{-\Im}}{\Gamma(1-\Im)}$.
- *If $u(x)$ is the constant function, then* $\frac{d^{\Im}(c)}{dx^{\Im}} = c\frac{x^{-\Im}}{\Gamma(1-\Im)}$.
- *The fractional derivative of $e^{\mu t}$ has the form,*

$$D^{\Im} e^{\mu t} = t^{-\Im} E_{1,1-\Im}(\mu t), \qquad n-1 < \Im < n, \qquad n \in \mathbb{N}, \qquad \mu \in \mathbb{C}.$$

Also, for the Caputo fractional derivative, we have

- *If $u(x)$ is the unit function, then* $\frac{d^{\Im}[1]}{dx^{\Im}} = 0$.
- *If $u(x)$ is the constant function, then* $\frac{d^{\Im}(c)}{dx^{\Im}} = 0$.
- *The fractional derivative of $e^{\mu t}$ has the form,*

$$_cD^{\Im} e^{\mu t} = \sum_{k=0}^{\infty} \frac{\mu^{k+n} t^{k+n-\Im}}{\Gamma(k+1+n-\Im)} = \mu^n t^{n-\Im} E_{1,n-\Im+1}(\mu t),$$

where $E_{1,n-\Im+1}$ is Mittag-leffler functions, $n-1 < \Im < n$, $n \in \mathbb{N}$, $\mu \in \mathbb{C}$.

3 $\mathbb{S}$-Transform

The $\mathbb{S}$- is a useful tool in applied and engineering mathematics, especially fractional differential equations. In this segment, we audit a few definitions and theorems related to the $\mathbb{S}$-transformation and its inverse [5, 22].

Definition 3.1. *The $\mathbb{S}$-transform of a given function $\mathrm{y(t)}$ is defined as,*

$$\mathbb{S}[\mathrm{y(t)}] = Y(s,\varpi) = \int_0^{\infty} \exp\left(-\frac{st}{\varpi}\right) \mathrm{y(t)} dt, \tag{6}$$

where $\mathbb{S}$, (s,ϖ), $\mathrm{y(t)}$ and $\exp\left(-\frac{st}{\varpi}\right)$ are operator of Suehu transform, positive integer, real function and kernal function, respectively.

Definition 3.2. *The inverse* $\mathbb{S}$*-transform is given by,*

$$\mathbb{S}^{-1}[Y(s,\varpi)] = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{1}{\varpi}\exp\left(\frac{st}{\varpi}\right) Y(s,\varpi)ds = \mathsf{y}(\mathsf{x}), \qquad x \geq 0. \tag{7}$$

Some of the useful $\mathbb{S}$-transform for some functions which are applied in this paper, defined by,

$$\begin{aligned}
\mathbb{S}[1] &= \frac{\varpi}{s},\\
\mathbb{S}[x] &= \frac{\varpi^2}{s^2},\\
\mathbb{S}[\sin(ax)] &= \frac{a\varpi^2}{s^2+a^2\varpi^2},\\
\mathbb{S}[\cos(ax)] &= \frac{s\varpi}{s^2+a^2\varpi^2},\\
\mathbb{S}[\exp(ax)] &= \frac{\varpi}{s-a\varpi},\\
\mathbb{S}[x^n] &= \left(\frac{\varpi}{s}\right)^{n+1} n! = \left(\frac{\varpi}{s}\right)^{n+1}\Gamma(n+1).
\end{aligned}$$

Theorem 3.1. *If* $\mathbb{S}[\mathsf{y}(\mathsf{x})] = Y(s,\varpi)$ *and* $\mathbb{S}[\mathsf{g}(\mathsf{x})] = G(s,\varpi)$*, the* $\mathbb{S}$*-transform of the function* $(\mathsf{y} * \mathsf{g})(\mathsf{x})$ *is defined as,*

$$\mathbb{S}[(\mathsf{y} * \mathsf{g})(\mathsf{x})] = Y(s,\varpi)G(s,\varpi). \tag{8}$$

Theorem 3.2. *If the function* $\mathsf{y}^{(\Im)}(\mathsf{x})$ *is the derivative of* $\mathsf{y}(\mathsf{x})$ *with respect to* x *then its* $\mathbb{S}$*-transform is defined by,*

$$\mathbb{S}[D^{\Im}\mathsf{y}(\mathsf{x})] = \left(\frac{s}{\varpi}\right)^{\Im} Y(s,\varpi) - \sum_{k=0}^{n-1}\left(\frac{s}{\varpi}\right)^{\Im-k-1} D^k\mathsf{y}(0). \tag{9}$$

4 Structural of The Approximate Solution in $2-$Dimensional for The Fractional Coupled Navier–Stokes Model

We can explain this method in two dimension, by considering the fractional coupled Navier–Stokes equation;

$$\begin{aligned}
D^{\Im}u + u\frac{\partial u}{\partial \zeta} + u\frac{\partial u}{\partial \xi} &= \rho\left(\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \xi^2}\right) + f_1,\\
D^{\Im}u + u\frac{\partial u}{\partial \zeta} + u\frac{\partial u}{\partial \xi} &= \rho\left(\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \xi^2}\right) + f_2,
\end{aligned} \tag{10}$$

for $n-1 < \Im \leq n$, $n \in \mathbf{N}$. With initial conditions,

$$\begin{aligned}
u(\zeta,\xi,0) &= u_0,\\
v(\zeta,\xi,0) &= v_0.
\end{aligned} \tag{11}$$

where, $D^{\Im}u$ and $D^{\Im}u$ are the Caputo fractional derivative of the functions u and u respectively, f_1 and f_2 are source term. In the first step, we will take the $\mathbb{S}$-transform to both sides to (10), as follows,

$$\begin{aligned} &\mathbb{S}\left[D^{\Im}u\right]+\mathbb{S}\left[u\frac{\partial u}{\partial\zeta}+u\frac{\partial u}{\partial\xi}\right]=\mathbb{S}\left[\rho\left(\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi^2}\right)\right]+\mathbb{S}\left[f_1\right],\\ &\mathbb{S}\left[D^{\Im}u\right]+\mathbb{S}\left[u\frac{\partial u}{\partial\zeta}+u\frac{\partial u}{\partial\xi}\right]=\mathbb{S}\left[\rho\left(\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi^2}\right)\right]+\mathbb{S}\left[f_2\right], \end{aligned} \tag{12}$$

or

$$\begin{aligned} &\mathbb{S}\left[D^{\Im}u\right]=\mathbb{S}\left[\rho\left(\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi^2}\right)\right]-\mathbb{S}\left[u\frac{\partial u}{\partial\zeta}+v\frac{\partial u}{\partial\xi}\right]+\mathbb{S}\left[f_1\right],\\ &\mathbb{S}\left[D^{\Im}u\right]=\mathbb{S}\left[\rho\left(\frac{\partial^2 v}{\partial\zeta^2}+\frac{\partial^2 v}{\partial\xi^2}\right)\right]-\mathbb{S}\left[u\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial\xi}\right]+\mathbb{S}\left[f_2\right], \end{aligned} \tag{13}$$

then,

$$\begin{aligned} \left(\frac{s}{\varpi}\right)^{\Im}\mathsf{U}(s,\varpi)&=\sum_{k=0}^{n-1}\left(\frac{s}{\varpi}\right)^{\Im-k-1}D^k u(\zeta,\xi,0)+\mathbb{S}\left[f_1\right]+\mathbb{S}\left[\rho\left(\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi^2}\right)\right]-\mathbb{S}\left[u\frac{\partial u}{\partial\zeta}+v\frac{\partial u}{\partial\xi}\right],\\ \left(\frac{s}{\varpi}\right)^{\Im}\mathsf{V}(s,\varpi)&=\sum_{k=0}^{n-1}\left(\frac{s}{\varpi}\right)^{\Im-k-1}D^k v(\zeta,\xi,0)+\mathbb{S}\left[f_2\right]+\mathbb{S}\left[\rho\left(\frac{\partial^2 v}{\partial\zeta^2}+\frac{\partial^2 v}{\partial\xi^2}\right)\right]-\mathbb{S}\left[u\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial\xi}\right]. \end{aligned} \tag{14}$$

Presently we multiply two sides of (14) by $\left(\frac{\varpi^{\Im}}{s^{\Im}}\right)$ to get

$$\begin{aligned} \mathsf{U}(s,\varpi)=&\sum_{k=0}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1}D^k u(\zeta,\xi,0)+\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_1\right]+\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi^2}\right)\right]\\ &-\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u\frac{\partial u}{\partial\zeta}+v\frac{\partial u}{\partial\xi}\right],\\ \mathsf{V}(s,\varpi)=&\sum_{k=0}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1}D^k v(\zeta,\xi,0)+\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_2\right]+\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2 v}{\partial\zeta^2}+\frac{\partial^2 v}{\partial\xi^2}\right)\right]\\ &-\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial\xi}\right]. \end{aligned} \tag{15}$$

By initial conditions (11) and taking inverse $\mathbb{S}$-transform for (15), we get

$$\begin{aligned} u=&\mathbb{S}^{-1}\left[\sum_{k=0}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1}D^k u_0\right]+\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_1\right]\right]+\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi^2}\right)\right]\right]\\ &-\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u\frac{\partial u}{\partial\zeta}+v\frac{\partial u}{\partial\xi}\right]\right], \end{aligned} \tag{16}$$

$$\begin{aligned} u=&\mathbb{S}^{-1}\left[\sum_{k=0}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1}D^k v_0\right]+\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_2\right]\right]+\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2 v}{\partial\zeta^2}+\frac{\partial^2 v}{\partial\xi^2}\right)\right]\right]\\ &-\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial\xi}\right]\right]. \end{aligned} \tag{17}$$

Now, we apply the Homotopy perturbation method [3, 10] to (16) and (17). In the beginning, we create the Homotopy such that,

$$\begin{aligned} u = u_0 + p\left[\mathbb{S}^{-1}\left[\sum_{k=1}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1} D^k u_0\right]\right] + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_1\right]\right]\right] \\ + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2 u}{\partial\zeta^2}+\frac{\partial^2 u}{\partial\xi^2}\right)\right]\right]\right] - p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u\frac{\partial u}{\partial\zeta}+v\frac{\partial u}{\partial\xi}\right]\right]\right], \end{aligned} \tag{18}$$

$$\begin{aligned} u = v_0 + p\left[\mathbb{S}^{-1}\left[\sum_{k=1}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1} D^k v_0\right]\right] + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_2\right]\right]\right] \\ + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2 v}{\partial\zeta^2}+\frac{\partial^2 v}{\partial\xi^2}\right)\right]\right]\right] - p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial\xi}\right]\right]\right], \end{aligned} \tag{19}$$

where $0 \leq p \leq 1$, p is the embedding parameter. The homotopy perturbation method involving decomposition the solution u and v in power series of the embedding parameter p by the form,

$$u = \sum_{n=0}^{\infty} u_n(\zeta,\xi,t)p^n, \tag{20}$$

$$u = \sum_{n=0}^{\infty} v_n(\zeta,\xi,t)p^n. \tag{21}$$

The non linear term can be decomposed as,

$$\left(u\frac{\partial u}{\partial\zeta}+v\frac{\partial u}{\partial\xi}\right) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{22}$$

$$\left(u\frac{\partial v}{\partial\zeta}+v\frac{\partial v}{\partial\xi}\right) = \sum_{n=0}^{\infty} p^n H_n(v). \tag{23}$$

Substituting (20) and (22) in (18), and (21) and (23) in (19), then we have

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(\zeta,\xi,t)p^n = u_0 + p\left[\mathbb{S}^{-1}\left[\sum_{k=1}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1} D^k u_0\right]\right] + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_1\right]\right]\right] \\ + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}\sum_{n=0}^{\infty} u_n(\zeta,\xi,t)p^n+\frac{\partial^2}{\partial\xi^2}\sum_{n=0}^{\infty} u_n(\zeta,\xi,t)p^n\right)\right]\right]\right] \\ - p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right], \end{aligned} \tag{24}$$

$$\begin{aligned} \sum_{n=0}^{\infty} v_n(\zeta,\xi,t)p^n = v_0 + p\left[\mathbb{S}^{-1}\left[\sum_{k=1}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1} D^k v_0\right]\right] + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_2\right]\right]\right] \\ + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}\sum_{n=0}^{\infty} v_n(\zeta,\xi,t)p^n+\frac{\partial^2}{\partial\xi^2}\sum_{n=0}^{\infty} v_n(\zeta,\xi,t)p^n\right)\right]\right]\right] \\ - p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right]. \end{aligned} \tag{25}$$

When $p = 1$ and equating the coefficient of power series of p , we get two collections of equations,

$$p^0 : u_0(\zeta,\xi,t) = u_0,$$

$$p^1 : u_1(\zeta,\xi,t) = \mathbb{S}^{-1}\left[\sum_{k=1}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1} D^k u_0\right] + \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_1\right]\right]$$
$$+ \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}u_0(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}u_0(\zeta,\xi,t)\right)\right]\right]$$
$$- \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_0(u)\right]\right],$$

$$p^2 : u_2(\zeta,\xi,t) = \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}u_1(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}u_1(\zeta,\xi,t)\right)\right]\right] - \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_1(u)\right]\right],$$

$$p^3 : u_3(\zeta,\xi,t) = \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}u_2(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}u_2(\zeta,\xi,t)\right)\right]\right] - \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_2(u)\right]\right],$$

$$\vdots$$

Also,

$$p^0 : v_0(\zeta,\xi,t) = v_0,$$

$$p^1 : v_1(\zeta,\xi,t) = \mathbb{S}^{-1}\left[\sum_{k=1}^{n-1}\left(\frac{s}{\varpi}\right)^{-k-1} D^k v_0\right] + \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[f_2\right]\right]$$
$$+ \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}v_0(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}v_0(\zeta,\xi,t)\right)\right]\right]$$
$$- \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_0(v)\right]\right],$$

$$p^2 : v_2(\zeta,\xi,t) = \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}v_1(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}v_1(\zeta,\xi,t)\right)\right]\right] - \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_1(v)\right]\right],$$

$$p^3 : v_3(\zeta,\xi,t) = \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}v_2(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}v_2(\zeta,\xi,t)\right)\right]\right] - \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_2(v)\right]\right],$$

$$\vdots$$

At last, we find the numerical approximate solution u and u by two series expansion,

$$u = \sum_{n=0}^{\infty} u_n(\zeta,\xi,t) = u_0(\zeta,\xi,t) + u_1(\zeta,\xi,t) + u_2(\zeta,\xi,t) + \dots, \tag{26}$$

$$u = \sum_{n=0}^{\infty} v_n(\zeta,\xi,t) = v_0(\zeta,\xi,t) + v_1(\zeta,\xi,t) + v_2(\zeta,\xi,t) + \dots. \tag{27}$$

5 Illustrative Examples

This segment is applied the strategy introduced in the paper and give solution of some fractional coupled Navier–Stokes model of $2-$dimensional.

Example 5.1. *Consider the fractional coupled Navier–Stokes equation of* $2-$*Dimensional,*

$$\begin{aligned} D^{\Im}u + u\frac{\partial u}{\partial \zeta} + v\frac{\partial u}{\partial \xi} &= \rho\left(\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \xi^2}\right) + f_1, \\ D^{\Im}v + u\frac{\partial v}{\partial \zeta} + v\frac{\partial v}{\partial \xi} &= \rho\left(\frac{\partial^2 v}{\partial \zeta^2} + \frac{\partial^2 v}{\partial \xi^2}\right) + f_2, \end{aligned} \tag{28}$$

with initial conditions,

$$\begin{aligned} u(\zeta,\xi,0) &= -\sin(\zeta+\xi), \\ v(\zeta,\xi,0) &= \sin(\zeta+\xi). \end{aligned} \tag{29}$$

When $\Im = 1$ *and* $(f_1 = f_2 = 0)$, (28) *has the exact solution,*

$$\begin{aligned} u &= -\exp(-2\rho t)\sin(\zeta+\xi), \\ u &= \exp(-2\rho t)\sin(\zeta+\xi). \end{aligned}$$

Solution: Using an approximation method that includes a combination of $\mathbb{S}$-transform and homotopy perturbation method, and explained in Section 4. We obtain the following models,

$$\begin{aligned} \sum_{n=0}^{\infty} u_n p^n = -\sin(\zeta+\xi) + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}\sum_{n=0}^{\infty}u_n(\zeta,\xi,t)p^n + \frac{\partial^2}{\partial\xi^2}\sum_{n=0}^{\infty}u_n(\zeta,\xi,t)p^n\right)\right]\right]\right] \\ - p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right], \end{aligned} \tag{30}$$

$$\begin{aligned} \sum_{n=0}^{\infty} v_n p^n = \sin(\zeta+\xi) + p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}\sum_{n=0}^{\infty}v_n(\zeta,\xi,t)p^n + \frac{\partial^2}{\partial\xi^2}\sum_{n=0}^{\infty}v_n(\zeta,\xi,t)p^n\right)\right]\right]\right] \\ - p\left[\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\sum_{n=0}^{\infty}p^n H_n(v)\right]\right]\right]. \end{aligned} \tag{31}$$

$$\begin{aligned} p^0 &: u_0(\zeta,\xi,t) = u_0 = -\sin(\zeta+\xi), \\ p^0 &: v_0(\zeta,\xi,t) = v_0 = \sin(\zeta+\xi), \\ p^1 &: u_1(\zeta,\xi,t) = \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}u_0(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}u_0(\zeta,\xi,t)\right)\right]\right] - \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_0(u)\right]\right] \\ &= \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}u_0(\zeta,\xi,t) + \frac{\partial^2}{\partial\xi^2}u_0(\zeta,\xi,t)\right)\right]\right] \\ &\quad - \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u_0\frac{\partial}{\partial\zeta}(u_0) + v_0\frac{\partial}{\partial\xi}(u_0)\right]\right] \\ &= 2\sin(\zeta+\xi)\frac{\rho t^{\Im}}{\Gamma(\Im+1)}, \end{aligned}$$

$$
\begin{aligned}
p^1 : v_1(\zeta,\xi,t) &= \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}v_0(\zeta,\xi,t)+\frac{\partial^2}{\partial\xi^2}v_0(\zeta,\xi,t)\right)\right]\right]-\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_0(v)\right]\right]\\
&= \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}v_0(\zeta,\xi,t)+\frac{\partial^2}{\partial\xi^2}v_0(\zeta,\xi,t)\right)\right]\right]\\
&\quad -\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u_0\frac{\partial}{\partial\zeta}(v_0)+v_0\frac{\partial}{\partial\xi}(v_0)\right]\right]\\
&= -2\sin(\zeta+\xi)\frac{\rho t^{\Im}}{\Gamma(\Im+1)},\\
p^2 : u_2(\zeta,\xi,t) &= \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}u_1(\zeta,\xi,t)+\frac{\partial^2}{\partial\xi^2}u_1(\zeta,\xi,t)\right)\right]\right]-\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_1(u)\right]\right]\\
&= \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}u_1(\zeta,\xi,t)+\frac{\partial^2}{\partial\xi^2}u_1(\zeta,\xi,t)\right)\right]\right]\\
&\quad -\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u_1\frac{\partial}{\partial\zeta}(u_0)+u_0\frac{\partial}{\partial\xi}(u_1)+v_1\frac{\partial}{\partial\zeta}(u_0)+v_0\frac{\partial}{\partial\xi}(u_1)\right]\right]\\
&= -4\sin(\zeta+\xi)\frac{\rho^2 t^{2\Im}}{\Gamma(2\Im+1)},\\
p^2 : v_2(\zeta,\xi,t) &= \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}v_1(\zeta,\xi,t)+\frac{\partial^2}{\partial\xi^2}v_1(\zeta,\xi,t)\right)\right]\right]-\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[H_1(v)\right]\right]\\
&= \mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[\rho\left(\frac{\partial^2}{\partial\zeta^2}v_1(\zeta,\xi,t)+\frac{\partial^2}{\partial\xi^2}v_1(\zeta,\xi,t)\right)\right]\right]\\
&\quad -\mathbb{S}^{-1}\left[\left(\frac{\varpi}{s}\right)^{\Im}\mathbb{S}\left[u_1\frac{\partial}{\partial\zeta}(v_0)+u_0\frac{\partial}{\partial\xi}(v_1)+v_1\frac{\partial}{\partial\zeta}(v_0)+v_0\frac{\partial}{\partial\xi}(v_1)\right]\right]\\
&= 4\sin(\zeta+\xi)\frac{\rho^2 t^{2\Im}}{\Gamma(2\Im+1)},\\
&\vdots
\end{aligned}
$$

Therefore the solution of (28) given by (26) and (27) as,

$$
\begin{aligned}
u &= u_0(\zeta,\xi,t)+u_1(\zeta,\xi,t)+u_2(\zeta,\xi,t)+\cdots\\
&= -\sin(\zeta+\xi)+2\sin(\zeta+\xi)\frac{\rho t^{\Im}}{\Gamma(\Im+1)}-4\sin(\zeta+\xi)\frac{\rho^2 t^{2\Im}}{\Gamma(2\Im+1)}+\cdots\\
&= \sin(\zeta+\xi)\left[-1+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\rho)^n t^{n\Im}}{\Gamma(n\Im+1)}\right].
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
u &= v_0(\zeta,\xi,t)+v_1(\zeta,\xi,t)+v_2(\zeta,\xi,t)+\cdots\\
&= \sin(\zeta+\xi)-2\sin(\zeta+\xi)\frac{\rho t^{\Im}}{\Gamma(\Im+1)}+4\sin(\zeta+\xi)\frac{\rho^2 t^{2\Im}}{\Gamma(2\Im+1)}+\cdots\\
&= \sin(\zeta+\xi)\left[1+\sum_{n=1}^{\infty}(-1)^{n}\frac{(2\rho)^n t^{n\Im}}{\Gamma(n\Im+1)}\right].
\end{aligned}
\tag{33}
$$

The numerical results of u and v at $\Im = 1$, are reported and compared between approximate solution and exact solution in Table 1, where show close contact with each other. In Table 2, for value of $\Im = 0.8$, we show compared the numerical results between iterative method [8] and the approximate solution.

Table 1: The numerical result for exact and approximate solution to the Example 5.1 $(\zeta = 0.001, \xi = 0.02, \rho = 1, \Im = 1)$.

t	Exact solution		Approximate solution	
	u	v	u	v
0.0	−0.021	0.021	−0.021	0.021
0.1	−0.017	0.017	−0.017	0.017
0.2	−0.014	0.014	−0.014	0.014
0.3	−0.012	0.012	−0.012	0.012
0.4	−0.009	0.009	−0.009	0.009
0.5	−0.008	0.008	−0.008	0.008
0.6	−0.006	0.006	−0.006	0.006
0.7	−0.005	0.005	−0.005	0.005
0.8	−0.004	0.004	−0.004	0.004
0.9	−0.003	0.003	−0.003	0.003

Table 2: The numerical result for iterative method and approximate solution to the Example 5.1 $(\zeta = 0.03, \xi = 0.1, \rho = 1, \Im = 0.8)$.

t	Iterative method [8]		Approximate solution	
	u	v	u	v
0.0	−0.130	0.130	−0.130	0.130
0.1	−0.105	0.093	−0.105	0.093
0.2	−0.086	0.074	−0.086	0.074
0.3	−0.071	0.061	−0.071	0.061
0.4	−0.059	0.052	−0.059	0.052
0.5	−0.050	0.044	−0.050	0.044
0.6	−0.043	0.039	−0.043	0.039
0.7	−0.037	0.034	−0.037	0.034
0.8	−0.032	0.030	−0.032	0.030
0.9	−0.028	0.027	−0.028	0.027

Example 5.2. *By referring to the fractional coupled Navier–Stokes for two-dimensional (28) in the previous example with initial conditions,*

$$\begin{aligned} u(\zeta, \xi, 0) &= -\exp(\zeta + \xi), \\ v(\zeta, \xi, 0) &= \exp(\zeta + \xi). \end{aligned} \tag{34}$$

The exact solution at $(\Im = 1, f_1 = f_2 = 0)$,

$$\begin{aligned} u &= -\exp(\zeta + \xi + 2\rho t), \\ u &= \exp(\zeta + \xi + 2\rho t). \end{aligned}$$

Solution: With the same method used in Section 4, which was applied in the previous example, we will obtain the approximate solution for both u and v, as in the result below,

$$\begin{aligned} u &= u_0(\zeta,\xi,t) + u_1(\zeta,\xi,t) + u_2(\zeta,\xi,t) + \cdots \\ &= -\exp(\zeta+\xi) + 2\exp(\zeta+\xi)\frac{\rho t^{\Im}}{\Gamma(\Im+1)} - 4\exp(\zeta+\xi)\frac{\rho^2 t^{2\Im}}{\Gamma(2\Im+1)} - \cdots \\ &= -\exp(\zeta+\xi)\left[1 + \sum_{n=1}^{\infty}\frac{(-2\rho)^n t^{n\Im}}{\Gamma(n\Im+1)}\right]. \end{aligned} \tag{35}$$

$$\begin{aligned} u &= v_0(\zeta,\xi,t) + v_1(\zeta,\xi,t) + v_2(\zeta,\xi,t) + \cdots \\ &= \exp(\zeta+\xi) - 2\exp(\zeta+\xi)\frac{\rho t^{\Im}}{\Gamma(\Im+1)} + 4\exp(\zeta+\xi)\frac{\rho^2 t^{2\Im}}{\Gamma(2\Im+1)} + \cdots \\ &= \exp(\zeta+\xi)\left[1 + \sum_{n=1}^{\infty}\frac{(-2\rho)^n t^{n\Im}}{\Gamma(n\Im+1)}\right]. \end{aligned} \tag{36}$$

Also the numerical results of u and v at $\Im = 1$, are reported and compared between approximate solution and exact solution in Table 3, we see exactly the same results. In Table 4, for value of $\Im = 0.8$, we show compared the numerical results between iterative method [8] and the approximate solution.

Table 3: The numerical result for exact and approximate solution to the Example 5.2 $(\zeta = 0.1, \xi = 0.01, \rho = 1, \Im = 1)$.

t	Exact solution		Approximate solution	
	u	v	u	v
0	-1.116	1.116	-1.116	1.116
0.1	-1.363	1.1363	-1.363	1.1363
0.2	-1.665	1.665	-1.665	1.665
0.3	-2.034	2.034	-2.034	2.034
0.4	-2.484	2.484	-2.484	2.484
0.5	-3.034	3.034	-3.034	3.034
0.6	-3.706	3.706	-3.706	3.706
0.7	-4.527	4.527	-4.527	4.527
0.8	-5.529	5.529	-5.529	5.529
0.9	-6.753	6.753	-6.753	6.753

Table 4: The numerical result for iterative method and approximate solution to the Example 5.2 ($\zeta = 0.1, \xi = 0.01, \rho = 1, \Im = 0.8$).

t	Iterative method [8]		Approximate solution	
	u	v	u	v
0	−1.116	1.116	−1.116	1.116
0.1	−0.804	0.804	−0.804	0.804
0.2	−0.641	0.641	−0.641	0.641
0.3	−0.529	0.529	−0.529	0.529
0.4	−0.446	0.446	−0.446	0.446
0.5	−0.383	0.383	−0.383	0.383
0.6	−0.333	0.333	−0.333	0.333
0.7	−0.294	0.294	−0.294	0.294
0.8	−0.261	0.261	−0.261	0.261
0.9	−0.234	0.234	−0.234	0.234

6 Conclusions

The main concern of the current paper is to examine the transformation of $\mathbb{S}$-transform and hybridize this transformation with the homogeneous perturbation method to solve FCNSE. This method played an important role in solving FCNSE. To illustrate this, some examples were combined to show the validity and applicability of this new technique and compare it with published works. Comparative tables were obtained about the efficiency of this work idea with other papers. The main technique considered in this paper can be used for fractional derivatives with order greater than 1, but the fractional differential equation requires more initial conditions to apply the proposed method, and we will try to expand and present it in the upcoming works. In addition, there are three directions for future work:

- Studying exact solution for system of FNSE.
- Applying other techniques for solving fractional differential equations as optimal control.
- Using the same technique by some transformation to solve another models of fractional partial differential equation.

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